Functional Equations for Landau–Ginzburg Models and Exponential Motives

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Context: Graph Potential QFTs

We adopt the framework developed by Belmans, Galkin, and Mukhopadhyay [7, 6, 5], which associates Laurent polynomials to trivalent colored graphs. These *graph potentials* are constructed by assigning to each trivalent vertex a Laurent polynomial in variables indexed by the incident edges, yielding a global potential by summing over all vertices.

For example, the vertex potential

$$Y(a,b,c) = \frac{a}{bc} + \frac{b}{ac} + \frac{c}{ab} + abc$$

satisfies the birational functional identity

$$Y(a,b,s) + Y(c,d,s) = Y(a,c,t) + Y(b,d,t) = Y(a,d,u) + Y(b,c,u),$$

with auxiliary variables (s,t,u) related by rational transformations preserving the volume. These identities govern combinatorial mutations of the graphs, producing equivalences between the resulting potentials.

Passing to the exponential

$$K(a,b,c) := \exp\bigl(Y(a,b,c)\bigr),$$

we obtain the expansion

$$K(a,b,c) = \sum_{i,j,k} F(i,j,k)\, a^i b^j c^k,$$

where the coefficients satisfy

$$\sum_{m,m'} F(i,j,m)\, g^{mm'}\, F(k,l,m') = \sum_{n,n'} F(i,k,n)\, g^{nn'}\, F(j,l,n'), \quad g^{mm'} = \delta_{m',-m}.$$

This structure is analogous to that of Frobenius algebras and two-dimensional topological quantum field theories (TQFTs).

Integration over internal variables gives partition functions

$$Z_\gamma(z_1,\ldots,z_n) = \int \exp\bigl(W_\gamma(z,w)\bigr)\, \frac{dw_1}{w_1} \cdots \frac{dw_{3g-3+n}}{w_{3g-3+n}},$$

depending only on the homotopy type of the graph γ and obeying gluing identities under decompositions along circles.

The main point of BGM proposal is that this structure is a shadow of a four-dimensional quantum field theory: the "usual" two-dimensional direction of mirror symmetry (Fukaya categories, quantum cohomology, and all counterparts) is complemented by two dimensions of model surfaces visualized by graphs: since Hatcher-Thurston trivalent graph is essentially same data as an oriented surface equipped with pants decomposition, and relations show independence on pants decompositions and 2d TQFT axioms. One notable difference between BGM 2d theory and simpler ones is that the Hilbert space of states of a circle is an infinite-dimensional Hilbert space of functions on this very circle, so some partition functions are inevitably undefined or infinite, e.g. partition function of a torus is the trace of the identity, which is the dimension of Hilbert space, infinity if taken literally.

Moreover, they point out that primary source of functional identities for exponential integrals is the geometric (birational) functional equation satisfied by Y. The passing from geometric entities as Y and W_γ to symbolic $K = \exp Y$, and then to the respective exponential integrals can be considered as first passing from geometric realm of LG models to a linearized realm of exponential motives (that lose much of geometry, but still keep some) and then to their homological, analytic and numerical realms of realizations. BGM and Kontsevich–Odesskii point out that there are multiple interpretations and realizations of the functional identities above.

Main aim of this project is to study those realizations that carry arithmetic information, such as Galois action, i.e. l-adic and crystalline realizations, and to try to make precise the BGM proposal above for them.

Similar framework is developed by Kontsevich and Odesskii under the name of "multiplication kernels".

Exponential motives and their realizations

Exponential integrals and D-modules

Let $W = W_{\gamma,c}$ be a Laurent polynomial in $n+k$ variables $x_1, \dots, x_n, q_1, \dots, q_k$ with integer coefficients.

The integral
$$G_W(t;q) = \int_{|x|=1} \exp\bigl(t\,W(x,q)\bigr)\,\frac{d\log x}{(2\pi i)^n},$$
converges absolutely and uniformly and defines function entire in t and holomorphic in $q_j \neq 0$.

Expanding the exponential into a power series, we obtain
$$G_W(t) = \sum_{k=0}^{\infty} \frac{c_k(W)}{k!}\,t^k,$$
where $c_k(W) = [W^k]_0$ denotes the constant term of the polynomial W^k. Note that $c_k(W)$ are Laurent polynomials in q with integer coefficients, and can be treated exactly on a computer.

The functional identities satisfied by a collection of functions W_γ result in explicit functional integral relations among the respective collection of exponential integrals, partition functions $Z_{g,n} = Z_\gamma = G_{W_\gamma}$.

Exponential sums and constructible sheaves

Let $\mathbb{F}_p^r$ be a finite field with p^r elements. Fix a nontrivial additive character

$$\psi : \mathbb{F}_{p^r} \to \mathbb{C}^\times, \quad \psi(x) = \exp\!\left(\frac{2\pi i \operatorname{Tr}_{\mathbb{F}_{p^r}/\mathbb{F}_p}(x)}{p}\right).$$

Consider a morphism of varieties over $\mathbb{F}_{p^r}$,

$$\mathrm{q} : Y \to Q,$$

together with a regular function

$$W : Y \to \mathbb{A}^1.$$

For each point $\bar{\mathrm{q}} \in Q(\mathbb{F}_{p^r})$, the fiber is

$$Y_{\bar{\mathrm{q}}} = \mathrm{q}^{-1}(\bar{\mathrm{q}}),$$

and we define the associated exponential sum by

$$S_W(\bar{\mathrm{q}}; q; \psi) = \sum_{y \in Y(\mathbb{F}_{p^r}) | \mathrm{q}(y) = \bar{\mathrm{q}}} \psi\big(W(y)\big).$$

A classical example is given by the Kloosterman sum. It arises by taking

$$Y = \big\{(x_1, x_2) \in (\mathbb{G}_m)^2\big\}, \quad Q = \mathbb{G}_m, \quad \mathrm{q} = x_1 x_2, \quad W = x_1 + x_2.$$

Then for each $\mathrm{q} \in \mathbb{F}_{p^r}^\times$,

$$S_W(\bar{\mathrm{q}}; q; \psi) = \sum_{\substack{x_1, x_2 \in \mathbb{F}_{p^r}^\times \\ x_1 x_2 = \bar{\mathrm{q}}}} \psi(x_1 + x_2) = \sum_{x \in \mathbb{F}_{p^r}^\times} \psi\!\left(x + \frac{\bar{\mathrm{q}}}{x}\right),$$

recovering the classical Kloosterman sum $\mathrm{Kl}_2(\bar{\mathrm{q}}; q; \psi)$.

By the Grothendieck–Lefschetz trace formula, this sum can be expressed as the trace function of the sheaf $\mathrm{q}_* W^* \mathcal{L}_\psi$ on Q where $\mathcal{L}_\psi$ denotes the Artin–Schreier sheaf associated to ψ.

Finally, the orthogonality relations for additive characters,

$$\sum_{x \in \mathbb{F}_{p^r}} \psi(tx) = \begin{cases} q & \text{if } t = 0, \\ 0 & \text{otherwise,} \end{cases}$$

is the counterpart of the delta pairing

$$g_{m,m'} = \delta_{m',-m}$$

which appears in the convolution identities of the complex theory.

Recent works [1, 8] have constructed motives from symmetric powers of Kloosterman sheaves, with Hodge numbers and L-functions aligning with predictions from mirror symmetry, revealing deep correspondences between monodromy spectra and Frobenius eigenvalues.

Functional equations

As we sketched out in the first section, the realizations of exponential motives of graph potentials in de Rham setting is studied in the papers of BGM and KO, where explicit formulas relating exponential integrals are proved. In contrast, the respective formulas for exponential sums were not yet proved, and their possible failure has not yet been checked. The aim of this project is to fill this gap, namely to compute some exponential sums of graph potentials, and to figure out what shall be the correction terms for the respective functional equations.

Research Goals

This project aims to:

1. Analyze how functional identities for graph potentials induce relations among exponential sums S_W over finite fields.
2. Identify and quantify corrective terms informed by weight filtrations or non-pure cohomology.
3. Formulate conjectural explanations via trace formulas for étale sheaves incorporating these corrections.
4. Investigate motivic and automorphic properties of the resulting exponential motives, with attention to connections to the Hodge theory of Kloosterman-type systems.

Methodology and Plan

The study integrates explicit computations and abstract theory through:

(1) Computational exploration of exponential sums and their discrepancies from complex predictions, using PARI/GP and SageMath.

(2) Extraction and formalization of corrective factors linked to geometric or cohomological phenomena.

(3) Conjectural formulation of trace identities for étale sheaves that incorporate these corrections.

(4) Rigorous proofs employing étale cohomology, weight arguments, and decomposition theorems to establish arithmetic analogues of the functional equations.

References

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