

Real and Complex Intersections of Opposite Open Schubert Cells

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Abstract

We compare the topology of the intersection of two opposite open Schubert cells in the real and complex complete flag varieties. In the real case, the number of connected components is highly nontrivial and was computed by Shapiro–Shapiro–Vainshtein. In the complex case, the corresponding space is an open Richardson variety, and Richardson’s theorem implies that it is irreducible, hence connected. We also briefly discuss the fundamental group of open Richardson varieties, and contrast it with what is currently known in the real setting.

1 Introduction

A basic problem in Schubert geometry is to understand the topology of intersections of opposite open cells. Over \mathbb{R} , this leads to a genuine connected-component problem. Over \mathbb{C} , the analogous incidence locus is irreducible, so the connected-component count becomes trivial and finer invariants, such as the fundamental group, become more interesting.

The purpose of this note is to place these two settings side by side. We first recall the real problem studied by Shapiro–Shapiro–Vainshtein [SSV97; SSV98], then explain its complex analogue in the language of open Richardson varieties, and finally mention what is known about the fundamental group in both settings.

2 The real setting

Let

$$\mathrm{Fl}_n(\mathbb{R}) \simeq \mathrm{SL}_n(\mathbb{R})/B(\mathbb{R})$$

be the variety of complete real flags, where $B(\mathbb{R})$ is the Borel subgroup of upper triangular matrices. Fix two opposite transversal flags f and g , and denote by

$$U_n^{f,g} \subset \mathrm{Fl}_n(\mathbb{R})$$

the set of all flags transversal to both f and g . Equivalently, $U_n^{f,g}$ is the intersection of two opposite open Schubert cells in $\mathrm{Fl}_n(\mathbb{R})$. Since $\mathrm{SL}_n(\mathbb{R})$ acts transitively on pairs of opposite transversal flags, the diffeomorphism type of $U_n^{f,g}$ is independent of the chosen pair, and we write simply U_n [SSV97].

A concrete affine model for this space is given in [SSV97]: one has

$$U_n \simeq N_n \setminus \Delta_n,$$

where N_n is the group of real unipotent upper triangular matrices and Δ_n is the union of certain determinantal divisors.

The main topological result in the real case is the following.

Theorem 2.1 (Shapiro–Shapiro–Vainshtein [SSV97; SSV98]). *The number of connected components of U_n is*

$$\#\pi_0(U_n) = \begin{cases} 2, & n = 2, \\ 6, & n = 3, \\ 20, & n = 4, \\ 52, & n = 5, \\ 3 \cdot 2^{n-1}, & n \geq 6. \end{cases}$$

More precisely, [SSV97] reduces the problem to a combinatorial one, and [SSV98] proves the stable formula $3 \cdot 2^{n-1}$.

3 The complex analogue

We now pass to the complex flag variety

$$\mathrm{Fl}_n(\mathbb{C}) \simeq \mathrm{SL}_n(\mathbb{C})/B(\mathbb{C}),$$

where $B(\mathbb{C})$ is the Borel subgroup of upper triangular matrices and $B^-(\mathbb{C})$ is the opposite Borel subgroup.

Let $W \simeq S_n$ be the Weyl group, and let $w_0 \in W$ be its longest element. In G/B , one has the Schubert cell

$$\mathring{X}^w := BwB/B$$

and the opposite Schubert cell

$$\mathring{X}_v := B^-vB/B.$$

Their intersection

$$\mathring{X}_v^w := \mathring{X}_v \cap \mathring{X}^w$$

is the open Richardson variety.

For the specific real problem above, the direct complex analogue is the intersection of the two opposite open big cells, namely

$$\mathring{X}_{\mathrm{id}}^{w_0} = (B^-B/B) \cap (Bw_0B/B) \subset \mathrm{SL}_n(\mathbb{C})/B(\mathbb{C}).$$

Thus the real space U_n should be viewed, on the complex side, as the open Richardson variety $\mathring{X}_{\mathrm{id}}^{w_0}$. More generally, arbitrary intersections of an opposite Schubert cell with a Schubert cell are exactly the open Richardson varieties \mathring{X}_v^w .

4 Irreducibility and connectedness in the complex case

The key input in the complex setting is Richardson’s theorem.

Theorem 4.1 (Richardson [Ric92]). *Let G be a connected reductive algebraic group over an algebraically closed field, let B be a Borel subgroup, and let B^- be the opposite Borel subgroup. If $v, w \in W$ satisfy $v \leq w$ in Bruhat order, then*

$$BwB \cap B^-vB$$

is a smooth irreducible locally closed subvariety of G .

This immediately yields the irreducibility of open Richardson varieties.

Proposition 4.2. *If $v \leq w$, then*

$$\hat{X}_v^w = (B^-vB/B) \cap (BwB/B)$$

is irreducible.

Proof. Let

$$\pi : G \longrightarrow G/B$$

be the quotient map. The subset $BwB \cap B^-vB \subset G$ is stable under right multiplication by B , and its image is exactly

$$\pi(BwB \cap B^-vB) = \hat{X}_v^w.$$

By Richardson's theorem, $BwB \cap B^-vB$ is irreducible when $v \leq w$. Since the image of an irreducible variety under a morphism is irreducible, \hat{X}_v^w is irreducible. \square

Remark 4.3 (A recent tilted analogue). A recent paper of Gao–Gao–Gao introduces *tilted Richardson varieties*

$$T_{u,v}, \quad T_{u,v}^\circ,$$

which are subvarieties of the complete flag variety indexed by arbitrary pairs of permutations $u, v \in S_n$. These varieties extend classical Richardson varieties: when $u \leq v$ in the usual Bruhat order, the tilted Richardson variety recovers the classical Richardson situation.

Their Theorem 1.4 proves a package of geometric properties for $T_{u,v}$ and $T_{u,v}^\circ$, including closedness/openness, a stratification by tilted Richardson strata, a dimension formula in terms of shortest paths in the quantum Bruhat graph, the closure relation

$$\overline{T_{u,v}^\circ} = T_{u,v},$$

and irreducibility of both $T_{u,v}^\circ$ and $T_{u,v}$ [GGG26].

The point relevant to the present note is not only the irreducibility statement itself, but also the method of proof. Gao–Gao–Gao emphasize that Richardson's original argument uses the structure of intersections of opposite Borel orbits. In the tilted setting there is no such direct Borel-orbit intersection description. To replace it, they introduce two new tools: tilted reduced words and a tilted analogue of the Deodhar decomposition.

More concretely, the proof of irreducibility proceeds by decomposing the open tilted Richardson variety $T_{u,v}^\circ$ into tilted Deodhar pieces. These pieces are indexed by tilted distinguished subwords of a regular tilted reduced word and are each of the form

$$\mathbb{C}^a \times (\mathbb{C}^*)^b.$$

Among these pieces there is a unique positive distinguished subword, which gives the unique cell of maximal dimension. Every irreducible component must lie in the closure of a maximal-dimensional

cell, and hence all components are forced to lie in the closure of this unique open piece. Therefore $T_{u,v}^\circ$ is irreducible, and then the closure relation implies irreducibility of $T_{u,v}$.

Thus, Gao–Gao–Gao provide a modern tilted analogue of Richardson-type irreducibility, but with a genuinely different proof: instead of relying on Borel-orbit intersections, the argument is combinatorial and cell-decomposition-theoretic, using tilted reduced words and tilted Deodhar decomposition.

Hence the connected-component problem in the complex case has a simple answer.

Theorem 4.4. *For every $v \leq w$, the open Richardson variety \mathring{X}_v^w is connected. In particular,*

$$\#\pi_0(\mathring{X}_v^w) = 1.$$

Therefore

$$\#\pi_0(\mathring{X}_{\text{id}}^{w_0}) = 1.$$

Proof. Every irreducible algebraic variety is connected in the Zariski topology, hence also connected in the usual complex topology. The claim follows from the previous proposition. \square

5 Fundamental groups

Since connectedness is automatic in the complex case, a more interesting topological invariant is the fundamental group.

5.1 The complex case

Recall that

$$\mathring{X}_v^u = \mathring{X}_v \cap \mathring{X}^u.$$

When $v \leq u$, this is an irreducible variety of dimension $\ell(u) - \ell(v)$; otherwise it is empty. In the special case of the longest Weyl group element w_0 , one has

$$\mathring{X}_v^{w_0} \simeq \mathring{X}_{\text{id}}^{w_0 v},$$

so questions about $\mathring{X}_v^{w_0}$ may often be reduced to the case $\mathring{X}_{\text{id}}^u$ [LSZ23].

The fundamental group of open Richardson varieties was studied by Li, Sottile, and Zhang in a setting motivated by mirror symmetry [LSZ23]. Let G be a complex simply-connected simple Lie group, let $P \supset B$ be a parabolic subgroup, and let w_P be the longest element of the Weyl group of the Levi subgroup of P . Then the open Richardson variety

$$\mathring{X}_{w_P}^{w_0}$$

is a log Calabi–Yau variety; in fact, it is isomorphic to the complement of the Knutson–Lam–Speyer anticanonical divisor in G/P [LSZ23].

In type A , their main theorem is the following.

Theorem 5.1 (Li–Sottile–Zhang [LSZ23]). *Let P be a parabolic subgroup of $\text{SL}(n, \mathbb{C})$. Then*

$$\pi_1\left(\mathring{X}_{w_P}^{w_0}\right) \cong \mathbb{Z}^{n-1}.$$

They also conjecture that if

$$\Gamma(v) := \{\alpha \in \Delta \mid s_\alpha \leq v\},$$

then

$$\pi_1(\mathring{X}_{\text{id}}^v) \cong \mathbb{Z}^{|\Gamma(v)|}.$$

This suggests that, in the complex setting, the fundamental group is a more informative invariant than the set of connected components.

5.2 The real counterpart

The real counterpart is presently much less uniform. In the real setting, the first classical invariant studied was the number of connected components, computed by Shapiro–Shapiro–Vainshtein [SSV97; SSV98]. Information about the fundamental group comes only indirectly, through results on the homotopy type of specific intersections.

The paper [AS23] shows that for $n \leq 4$, every connected component of every BL_σ is contractible; hence in these cases the fundamental group is trivial. The same paper also exhibits an example with $\sigma = [563412] \in S_6$ for which some connected components are homotopy equivalent to S^1 , and therefore have fundamental group \mathbb{Z} . Moreover, for $n \geq 5$ and the top permutation, there are noncontractible connected components, although that statement alone does not determine their fundamental group in general [AS23].

The companion paper [AS21] studies further examples, especially the case $n = 4$, and again finds that all connected components are contractible. More recently, the thesis [Lui25] studies intersections of real Bruhat cells in dimension 6 and proves that for permutations in S_6 with at most 12 inversions, all connected components are contractible except for the same exceptional permutation [563412], where one finds components of homotopy type S^1 .

Thus, unlike the complex setting, where a clean general theorem is available for the fundamental group in the mirror-symmetric case, the real counterpart is currently understood mainly through case-by-case homotopy computations: the known fundamental groups are either trivial or \mathbb{Z} , depending on whether the corresponding component is contractible or homotopy equivalent to a circle.

6 Conclusion

The real and complex settings are formally parallel but topologically quite different.

In the real flag variety, the intersection of two opposite open Schubert cells,

$$U_n \subset \text{SL}_n(\mathbb{R})/B(\mathbb{R}),$$

may have many connected components, and their enumeration is a subtle problem solved by Shapiro–Shapiro–Vainshtein [SSV97; SSV98].

In the complex flag variety, the corresponding incidence locus is the open Richardson variety

$$\mathring{X}_{\text{id}}^{w_0} \subset \text{SL}_n(\mathbb{C})/B(\mathbb{C}),$$

and more generally \mathring{X}_v^w . By Richardson’s theorem [Ric92], these varieties are irreducible whenever nonempty, hence connected. Therefore

$$\#\pi_0(\mathring{X}_v^w) = 1.$$

The natural topological question in the complex setting is therefore not the number of connected components, but rather finer invariants such as the fundamental group.

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