EXCEPTIONAL COLLECTIONS FOR CANONICAL STACKS OF LOG DEL PEZZO SURFACES WITH $\frac{1}{3}(1,1)$ SINGULARITIES

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Abstract. We study derived categories associated with log del Pezzo surfaces whose singularities are of type $\frac{1}{3}(1,1)$. For such a surface X, we consider the canonical smooth Deligne–Mumford stack $\pi : \mathcal{X} \to X$ and also discuss the singular coarse surface X.

Our main result proves that, if X is a complex log del Pezzo surface whose singularities are all of type $\frac{1}{3}(1,1)$, then $D^b(\operatorname{coh} \mathcal{X})$ admits a full exceptional collection. We also give an explicit description of the local exceptional objects supported on the residual gerbes of the stacky points.

As an application, we study a general degree 10 hypersurface $X_{10} \subset \mathbb{P}(1,2,3,5)$, one of the sporadic Johnson–Kollár examples. We show that its canonical stack $\mathcal{X}_{10}$ has a full exceptional collection of length 13, and we discuss the corresponding singular coarse category.

1. Introduction

Let X be a log del Pezzo surface over $\mathbb{C}$. Thus X is a normal projective surface with klt singularities and ample anticanonical divisor $-K_X$. In dimension two, klt singularities are precisely quotient singularities. In this paper we focus on the case where all singularities are cyclic quotient singularities of type $\frac{1}{3}(1,1)$.

There are two natural categorical objects attached to such a surface. The first one is the derived category of the canonical smooth Deligne–Mumford stack $\pi : \mathcal{X} \to X$. This stack agrees with X over the smooth locus and remembers the stabilizer groups at the quotient singularities. The second one is the derived category $D^b(\operatorname{coh} X)$ of the singular coarse surface. These categories are related, but they are not the same: the stack is smooth, while the coarse surface is singular.

This point of view is natural in the study of log del Pezzo surfaces in weighted projective spaces. Johnson–Kollár classified anticanonically embedded quasismooth log del Pezzo hypersurfaces in weighted projective 3-spaces [10]. Later work on exceptional collections for such surfaces includes

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Elagin's construction for a family of non-toric log-terminal del Pezzo hypersurfaces, treated as smooth stacks [6], and the work of Gugiatti–Rota on full exceptional collections for canonical stacks associated with the main Johnson–Kollár series [7].

The first purpose of this paper is to record a useful categorical consequence of the special McKay correspondence in the log del Pezzo setting. For surfaces whose singularities are all of type $\frac{1}{3}(1,1)$, the local Ishii–Ueda complement is especially simple: each singular point contributes exactly one gerbe-supported exceptional object. Since the minimal resolution of a log del Pezzo surface is rational, this gives a full exceptional collection on the canonical stack, with an explicit length formula.

The second purpose is to apply this statement to a sporadic example in the Johnson–Kollár list. The surface

$$X_{10} \subset \mathbb{P}(1,2,3,5)$$

is not one of the main infinite series treated in [7]. We treat this example by using the Corti–Heuberger cascade, which identifies its minimal resolution as a blow-up of the Hirzebruch surface $\mathbb{F}_3$. This gives an explicit categorical construction for the canonical stack of X_{10}, and it also allows us to discuss the singular coarse category through the approach of Karmazyn–Kuznetsov–Shinder.

The main result of the paper is the following.

Theorem 1.1. *Let X be a log del Pezzo surface whose singularities are all of type $\frac{1}{3}(1,1)$, and let $\pi : \mathcal{X} \to X$ be its canonical stack. Suppose that the singular points of X are $p_1, \ldots, p_r$. Then $D^b(\operatorname{coh} \mathcal{X})$ admits a full exceptional collection.*

More precisely, if $f : \widetilde{X} \to X$ is the minimal resolution, then there is a semiorthogonal decomposition

$$D^b(\operatorname{coh} \mathcal{X}) = \left\langle \mathcal{E}_{p_1}, \ldots, \mathcal{E}_{p_r}, \Phi(D^b(\operatorname{coh} \widetilde{X})) \right\rangle,$$

where $\mathcal{E}_{p_i}$ is the gerbe-supported exceptional object corresponding to the unique non-special representation of the stabilizer $\boldsymbol{\mu}_3$ at p_i. Moreover,

$$\operatorname{rk} K_0(\widetilde{X}) = 12 - K_X^2 + \frac{r}{3},$$

and therefore $D^b(\operatorname{coh} \mathcal{X})$ has a full exceptional collection of length

$$12 - K_X^2 + \frac{4r}{3}.$$

We then specialize to the weighted hypersurface

$$X_{10} = V(F_{10}) \subset \mathbb{P}(1,2,3,5).$$

For a general quasismooth well-formed member, X_{10} is a log del Pezzo surface with a unique singularity of type $\frac{1}{3}(1,1)$ and $K_{X_{10}}^2 = \frac{1}{3}$. Hence X_{10} belongs to the family $X_{1,1/3}$ classified by Corti and Heuberger, and their cascade construction identifies its minimal resolution as

$$\widetilde{X}_{10} \cong \mathrm{Bl}_8 \mathbb{F}_3.$$

This is the geometric input needed to make the general theorem explicit in the sporadic case.

Theorem 1.2. *Let $X_{10} \subset \mathbb{P}(1,2,3,5)$ be a general degree 10 hypersurface, and let $\pi : \mathcal{X}_{10} \to X_{10}$ be its canonical stack. Then $D^b(\operatorname{coh} \mathcal{X}_{10})$ has a full exceptional collection of length 13.*

We also record the corresponding decomposition for the singular coarse surface. The approach of Karmazyn–Kuznetsov–Shinder descends compatible semiorthogonal decompositions from a resolution to a surface with rational singularities. In the model case $\mathbb{F}_3 \to \mathbb{P}(1,1,3)$, their construction produces a local component equivalent to the derived category of modules over the Kalck–Karmazyn algebra

$$K(3,1) \cong k[z_1, z_2]/(z_1, z_2)^2.$$

Using the Corti–Heuberger cascade and Orlov's blow-up formula, we obtain a compatible decomposition on the minimal resolution of X_{10}, and hence a semiorthogonal decomposition

$$D^b(\operatorname{coh} X_{10}) = \left\langle D^b(K(3,1)\text{-mod}), F_1, \ldots, F_{10} \right\rangle,$$

where $F_1, \ldots, F_{10}$ are exceptional objects.

The paper is organized as follows. In Section 2 we recall rationality of log del Pezzo surfaces and explain why the minimal resolution admits a full exceptional collection. In Section 3 we recall canonical stacks, the local structure of the singularity $\frac{1}{3}(1,1)$, and prove the general full exceptional collection theorem. In Section 4 we compare this stack-theoretic construction with the approach of Karmazyn–Kuznetsov–Shinder to the singular category $D^b(\operatorname{coh} X)$. In Section 5 we use the Corti–Heuberger cascade to identify the minimal resolution of X_{10} and derive the categorical consequences for both the canonical stack and the singular coarse surface. Finally, in Section 6 we discuss how cascades and good models suggest similar constructions in broader families.

2. Resolutions and exceptional collections

In this work, we consider only algebraic varieties defined over the field $\mathbb{C}$ of complex numbers. We will not mention this assumption again. Most of the background material on log del Pezzo surfaces used in this section can be found in Alexeev–Nikulin [1], while the results and perspective on exceptional sequences on smooth rational surfaces are closely related to the work of Hille–Perling [8]. We recall the arguments that are needed later in order to keep the paper self-contained.

Definition 2.1. A *log del Pezzo surface* is a normal projective surface X with klt singularities such that $-K_X$ is ample.

For normal surfaces, klt singularities are precisely quotient singularities [13, Corollary 5.21]. Moreover, klt surface singularities are rational [13, Theorem 5.22]. Hence, if $f : \widetilde{X} \to X$ is any resolution, then $f_*\mathcal{O}_{\widetilde{X}} = \mathcal{O}_X$ and $R^i f_*\mathcal{O}_{\widetilde{X}} = 0$ for all $i > 0$. Consequently, the Leray spectral sequence gives a canonical isomorphism

$$H^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) \cong H^1(X, \mathcal{O}_X).$$

We first record the rationality property that will allow us to apply Orlov's blow-up formula to the minimal resolution.

Proposition 2.2. *Let X be a log del Pezzo surface, and let $f : \widetilde{X} \to X$ be its minimal resolution. Then $\widetilde{X}$ is a smooth rational surface. In particular, X is rational.*

Proof. Since X has klt singularities and $-K_X$ is ample, Kawamata–Viehweg vanishing gives $H^1(X, \mathcal{O}_X) = 0$. Since the singularities of X are rational, the Leray isomorphism above gives $H^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) = 0$.

It remains to show that $H^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(2K_{\widetilde{X}})) = 0$. Suppose, by contradiction, that there exists an effective divisor $D \sim 2K_{\widetilde{X}}$. The pullback $-f^*K_X$ is nef and big, because $-K_X$ is ample. Hence $(-f^*K_X) \cdot D \geq 0$. On the other hand, if

$$K_{\widetilde{X}} = f^*K_X + \sum_i a_i E_i$$

is the discrepancy formula, then $f^*K_X \cdot E_i = 0$ for every exceptional curve E_i. Therefore

$$(-f^*K_X) \cdot D = (-f^*K_X) \cdot 2K_{\widetilde{X}} = -2K_X^2 < 0,$$

because $(-K_X)^2 > 0$. This contradiction proves that $H^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(2K_{\widetilde{X}})) = 0$. By Castelnuovo's rationality criterion, $\widetilde{X}$ is rational. Since X is birational to $\widetilde{X}$, the surface X is rational as well. □

Example 2.3. The weighted projective plane $\mathbb{P}(1, 1, 3)$ is a log del Pezzo surface with a unique singularity of type $\frac{1}{3}(1, 1)$. Its minimal resolution is the Hirzebruch surface $\mathbb{F}_3 = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(3))$, and the exceptional curve is the negative section $C_0 \subset \mathbb{F}_3$ with $C_0^2 = -3$. See, for example, [12, Example 4.2.1].

We now recall why every smooth projective rational surface admits a full exceptional collection.

Definition 2.4. Let $\mathcal{T}$ be a triangulated category. An object $E \in \mathcal{T}$ is *exceptional* if $\mathrm{Hom}(E, E) = \mathbb{C}$ and $\mathrm{Ext}^k(E, E) = 0$ for all $k \neq 0$. A sequence $\langle E_1, \ldots, E_n \rangle$ is an *exceptional collection* if each object E_i is exceptional and $\mathrm{Ext}^\bullet(E_j, E_i) = 0$ for all $j > i$. The collection is *full* if it generates the triangulated category.

The projective plane has Beilinson's full exceptional collection [2]

$$D^b(\operatorname{coh}\mathbb{P}^2) = \langle \mathcal{O}, \mathcal{O}(1), \mathcal{O}(2)\rangle .$$

Similarly, every Hirzebruch surface $\mathbb{F}_n = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(n))$ admits a standard full exceptional collection. Let C_0 denote the negative section, and let f denote the fiber class. With the convention $C_0^2 = -n$, $C_0 \cdot f = 1$, and $f^2 = 0$, one such collection is

$$\langle \mathcal{O}, \mathcal{O}(f), \mathcal{O}(C_0 + nf), \mathcal{O}(C_0 + (n+1)f)\rangle .$$

We will use the following point blow-up case of Orlov's theorem on monoidal transformations [14, Theorem 4.3 and Corollary 4.4].

Theorem 2.5 (Orlov's blow-up formula)**.** *Let S be a smooth projective surface, and let $\beta : \operatorname{Bl}_p S \to S$ be the blow-up of S at a point p, with exceptional divisor E. Then there is a semiorthogonal decomposition*

$$D^b(\operatorname{coh}\operatorname{Bl}_p S) = \Big\langle \mathcal{O}_E(-1), \beta^* D^b(\operatorname{coh} S)\Big\rangle ,$$

where $\mathcal{O}_E(-1)$ is regarded as an object of $D^b(\operatorname{coh}\operatorname{Bl}_p S)$ via the inclusion $E \hookrightarrow \operatorname{Bl}_p S$. In particular, if $D^b(\operatorname{coh} S)$ admits a full exceptional collection, then $D^b(\operatorname{coh}\operatorname{Bl}_p S)$ also admits a full exceptional collection.

Corollary 2.6. *Every smooth projective rational surface admits a full exceptional collection.*

Proof. Every smooth projective rational surface is obtained from either $\mathbb{P}^2$ or a Hirzebruch surface $\mathbb{F}_n$ by a sequence of blow-ups at points. The surfaces $\mathbb{P}^2$ and $\mathbb{F}_n$ have full exceptional collections. By Theorem 2.5, each point blow-up preserves the existence of a full exceptional collection. Hence every smooth projective rational surface admits a full exceptional collection. □

Combining Proposition 2.2 and Corollary 2.6, we obtain the following consequence.

Corollary 2.7. *Let X be a log del Pezzo surface, and let $f : \widetilde{X} \to X$ be its minimal resolution. Then $D^b(\operatorname{coh}\widetilde{X})$ admits a full exceptional collection.*

3. Canonical stacks and exceptional collections

We now introduce the stack-theoretic replacement of a singular surface and recall the local representation-theoretic input needed for the semiorthogonal decomposition. We then prove the general full exceptional collection theorem for canonical stacks of log del Pezzo surfaces with $\frac{1}{3}(1,1)$ singularities.

Definition 3.1. Let X be a normal surface with quotient singularities. The *canonical stack* of X is the smooth Deligne–Mumford stack $\pi : \mathcal{X} \to X$ whose coarse moduli space is X, whose local models at quotient singularities are the corresponding quotient stacks, and whose structure morphism is an isomorphism over the smooth locus of X.

At a singular point of type $\frac{1}{3}(1,1)$, the local analytic model is $\mathbb{A}^2/\boldsymbol{\mu}_3$, where $\boldsymbol{\mu}_3$ acts by $\zeta \cdot (x,y) = (\zeta x, \zeta y)$. Hence the canonical stack is locally $[\mathbb{A}^2/\boldsymbol{\mu}_3]$. Moreover, the minimal resolution of this singularity is determined by the Hirzebruch–Jung continued fraction $\frac{3}{1} = [3]$, so the exceptional locus consists of one smooth rational curve E with $E^2 = -3$.

We recall the local terminology used in the special McKay correspondence. Let $G \subset \mathrm{GL}_2(\mathbb{C})$ be a finite small subgroup, let $R = \mathbb{C}[x,y]$, and let

$$\nu : Y \longrightarrow \mathrm{Spec} R^G$$

be the minimal resolution of the quotient singularity. For an irreducible representation ρ of G, set

$$M_\rho := (R \otimes \rho^\vee)^G.$$

This is a reflexive R^G-module. The associated *full sheaf* on Y is

$$\mathcal{M}_\rho := \nu^* M_\rho/\mathrm{torsion}.$$

Wunram's correspondence associates indecomposable full sheaves on Y with irreducible representations of G. A full sheaf $\mathcal{M}_\rho$ is called *special* if

$$H^1(Y, \mathcal{M}_\rho^\vee) = 0.$$

Equivalently, the corresponding representation ρ is called a *special representation*. A representation is called *non-special* if it is not special; see [15] and also [9, Section 2].

We now determine the non-special representations for the singularity $\frac{1}{3}(1,1)$. Let ρ_0, ρ_1, ρ_2 be the irreducible characters of $\boldsymbol{\mu}_3$, indexed by $\rho_i(\zeta) = \zeta^i$. In the cyclic case $\frac{1}{n}(1,q)$, the special representations can be read from the sequence associated with the Hirzebruch–Jung continued fraction; see [9, Theorem 3.2]. In our case, $n = 3$, $q = 1$, and $\frac{3}{1} = [3]$, so the associated sequence is

$$i_0 = 3, \qquad i_1 = 1, \qquad i_2 = 0.$$

Thus the special representations are $\rho_{i_0} = \rho_0$ and $\rho_{i_1} = \rho_1$. Since the irreducible characters of $\boldsymbol{\mu}_3$ are exactly ρ_0, ρ_1, ρ_2, the unique non-special representation is ρ_2.

For a singular point $p \in X$ of type $\frac{1}{3}(1,1)$, the corresponding local exceptional object can be written explicitly. Let

$$\iota_p : B\boldsymbol{\mu}_3 \hookrightarrow \mathcal{X}$$

be the residual gerbe over p. We denote by the same symbol ρ_2 the line bundle on $B\boldsymbol{\mu}_3$ associated with the character $\rho_2(\zeta) = \zeta^2$. We define

$$\mathcal{E}_p := \iota_{p*}(\mathcal{O}_{B\boldsymbol{\mu}_3} \otimes \rho_2).$$

Equivalently, after identifying an étale neighbourhood of p with the quotient stack $[\mathbb{A}^2/\boldsymbol{\mu}_3]$, the object $\mathcal{E}_p$ is represented by the $\boldsymbol{\mu}_3$-equivariant module

$$\mathbb{C}[x,y]/(x,y) \otimes \rho_2.$$

Thus x and y act by zero, and the stabilizer acts on the one-dimensional fiber by the character ρ_2. In the local notation of Ishii–Ueda, this is the object $\mathcal{O}_0 \otimes \rho_2$.

Theorem 3.2 (Ishii–Ueda, [9, Theorem 1.4])**.** *Let X be a surface with at worst quotient singularities, let $\mathcal{X}$ be its canonical stack, and let $Y \to X$ be the minimal resolution. Then there is a fully faithful functor*

$$\Phi : D^b(\operatorname{coh} Y) \to D^b(\operatorname{coh} \mathcal{X})$$

and a semiorthogonal decomposition

$$D^b(\operatorname{coh} \mathcal{X}) = \left\langle E_1, \dots, E_\ell, \Phi(D^b(\operatorname{coh} Y)) \right\rangle,$$

where $E_1, \dots, E_\ell$ is an exceptional collection.

In the local cyclic case, Ishii–Ueda identify the semiorthogonal complement through the non-special representations; see [9, Proposition 1.1 and Theorem 1.2]. Hence a singularity of type $\frac{1}{3}(1,1)$ contributes exactly one exceptional object to the complement, namely the object $\mathcal{E}_p$ defined above.

We now prove the main theorem for canonical stacks.

Theorem 3.3. *Let X be a log del Pezzo surface whose singularities are all of type $\frac{1}{3}(1,1)$. Let $\pi : \mathcal{X} \to X$ be the canonical stack, and let $f : \widetilde{X} \to X$ be the minimal resolution. Suppose that the singular points of X are $p_1, \dots, p_r$. Then there is a semiorthogonal decomposition*

$$D^b(\operatorname{coh} \mathcal{X}) = \left\langle \mathcal{E}_{p_1}, \dots, \mathcal{E}_{p_r}, \Phi(D^b(\operatorname{coh} \widetilde{X})) \right\rangle.$$

In particular, $D^b(\operatorname{coh} \mathcal{X})$ admits a full exceptional collection.

Proof. By Proposition 2.2, the minimal resolution $\widetilde{X}$ is a smooth rational surface. Therefore $D^b(\operatorname{coh} \widetilde{X})$ admits a full exceptional collection by Corollary 2.6.

By Theorem 3.2, there is a fully faithful functor

$$\Phi : D^b(\operatorname{coh} \widetilde{X}) \hookrightarrow D^b(\operatorname{coh} \mathcal{X})$$

and a semiorthogonal decomposition whose complement is generated by the local exceptional objects associated with the non-special representations. Since each singular point p_i has type $\frac{1}{3}(1,1)$, it has a unique non-special representation, namely ρ_2. Thus the local object attached to p_i is

$$\mathcal{E}_{p_i} = \iota_{p_i *}(\mathcal{O}_{B\boldsymbol{\mu}_3} \otimes \rho_2),$$

and the decomposition becomes

$$D^b(\operatorname{coh} \mathcal{X}) = \left\langle \mathcal{E}_{p_1}, \dots, \mathcal{E}_{p_r}, \Phi(D^b(\operatorname{coh} \widetilde{X})) \right\rangle.$$

The objects $\mathcal{E}_{p_i}$ have disjoint supports for distinct p_i, so they are mutually orthogonal. Since $D^b(\operatorname{coh} \widetilde{X})$ has a full exceptional collection and the remaining components are generated by the exceptional objects $\mathcal{E}_{p_i}$, the category $D^b(\operatorname{coh} \mathcal{X})$ has a full exceptional collection. □

Corollary 3.4. *With the notation of Theorem 3.3, the category* $D^b(\operatorname{coh}\mathcal{X})$ *has a full exceptional collection of length*

$$12 - K_X^2 + \frac{4r}{3}.$$

Equivalently, the minimal resolution satisfies

$$\operatorname{rk} K_0(\widetilde{X}) = 12 - K_X^2 + \frac{r}{3},$$

and the canonical stack adds one further exceptional object for each singular point.

Proof. Let $E_i \subset \widetilde{X}$ be the exceptional curve over p_i. Since each singularity has type $\frac{1}{3}(1,1)$, we have $E_i^2 = -3$, and the discrepancy formula is

$$K_{\widetilde{X}} = f^*K_X - \frac{1}{3}\sum_{i=1}^{r} E_i.$$

The curves E_i are disjoint, and $f^*K_X \cdot E_i = 0$ for all i. Hence, we have

$$K_{\widetilde{X}}^2 = K_X^2 + \frac{1}{9}\sum_{i=1}^{r} E_i^2 = K_X^2 - \frac{r}{3}.$$

Since $\widetilde{X}$ is a smooth rational surface, Noether's formula gives $\rho(\widetilde{X}) = 10 - K_{\widetilde{X}}^2$. Therefore, we obtain

$$\operatorname{rk} K_0(\widetilde{X}) = 2 + \rho(\widetilde{X}) = 12 - K_{\widetilde{X}}^2 = 12 - K_X^2 + \frac{r}{3}.$$

Finally, the semiorthogonal decomposition in Theorem 3.3 adds one exceptional object $\mathcal{E}_{p_i}$ for each singular point p_i. Thus the full exceptional collection on $D^b(\operatorname{coh}\mathcal{X})$ has length

$$\left(12 - K_X^2 + \frac{r}{3}\right) + r = 12 - K_X^2 + \frac{4r}{3}. \qquad \square$$

4. The singular coarse surface and the Kuznetsov approach

We now turn from the canonical stack to the singular coarse surface. The two categories $D^b(\operatorname{coh}\mathcal{X})$ and $D^b(\operatorname{coh} X)$ should not be confused: the stack $\mathcal{X}$ is smooth, whereas the coarse surface X is singular. For $D^b(\operatorname{coh} X)$, we use the approach of Karmazyn–Kuznetsov–Shinder to derived categories of surfaces with rational singularities.

Let $f : \widetilde{X} \to X$ be a resolution. The method of Karmazyn–Kuznetsov–Shinder starts with a semiorthogonal decomposition of $D^b(\operatorname{coh}\widetilde{X})$ and asks when it descends to a semiorthogonal decomposition of $D^b(\operatorname{coh} X)$. The relevant compatibility condition requires the objects $\mathcal{O}_E(-1)$, where E runs through the irreducible exceptional curves of f, to belong to the appropriate components of the semiorthogonal decomposition on $\widetilde{X}$; see [11, Definition 2.7 and Theorem 2.12]. Under this condition, the pushforward functor induces a semiorthogonal decomposition of $D^b(\operatorname{coh} X)$.

The basic local model for us is the contraction

$$\mathbb{F}_3 \longrightarrow \mathbb{P}(1,1,3),$$

which contracts the negative section $E \subset \mathbb{F}_3$ with $E^2 = -3$. Karmazyn–Kuznetsov–Shinder treat the more general contraction $\mathbb{F}_d \to \mathbb{P}(1,1,d)$. In their notation, if H denotes the pullback of the point class from $\mathbb{P}^1$, then they use the full exceptional collection

$$D^b(\operatorname{coh}\mathbb{F}_d) = \langle \mathcal{O}(-H-E), \mathcal{O}(-H), \mathcal{O}, \mathcal{O}(C) \rangle ,$$

where E is the exceptional curve and C is the disjoint section with $C^2 = d$. They group this collection as

$$\widetilde{\mathcal{A}}_1 = \langle \mathcal{O}(-H-E), \mathcal{O}(-H) \rangle, \qquad \widetilde{\mathcal{A}}_2 = \langle \mathcal{O} \rangle, \qquad \widetilde{\mathcal{A}}_3 = \langle \mathcal{O}(C) \rangle.$$

The decomposition is compatible with the contraction because $\mathcal{O}_E(-1)$ belongs to $\widetilde{\mathcal{A}}_1$. Therefore it descends to

$$D^b(\operatorname{coh}\mathbb{P}(1,1,d)) = \langle \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3 \rangle ,$$

where $\mathcal{A}_i = f_*(\widetilde{\mathcal{A}}_i)$. Moreover,

$$\mathcal{A}_1 \simeq D^b(K(d,1)\text{-mod}), \qquad \mathcal{A}_2 \simeq D^b(k), \qquad \mathcal{A}_3 \simeq D^b(k);$$

see [11, Example 3.17]. For $d = 3$, this gives

$$D^b(\operatorname{coh}\mathbb{P}(1,1,3)) = \left\langle D^b(K(3,1)\text{-mod}), D^b(k), D^b(k) \right\rangle .$$

The algebra $K(3,1)$ is the Kalck–Karmazyn algebra associated with the cyclic quotient singularity $\frac{1}{3}(1,1)$, and in this case

$$K(3,1) \cong k[z_1,z_2]/(z_1,z_2)^2;$$

see [11, Example 3.14].

We also record that the Brauer obstruction does not appear in the examples considered here. We use Bright's exact sequence for a normal surface X with rational singularities. If $f : Y \to X$ is the minimal resolution and $E \subset \operatorname{Pic}(Y)$ denotes the subgroup generated by the exceptional curves, then the intersection pairing induces a homomorphism $\theta : \operatorname{Pic}(Y) \to E^*$, and there is an exact sequence

$$0 \longrightarrow \operatorname{Pic}(X) \longrightarrow \operatorname{Pic}(Y) \xrightarrow{\theta} E^* \longrightarrow \operatorname{Br}(X) \longrightarrow \operatorname{Br}(Y);$$

see [3, Proposition 1]. Since the resolutions considered below are rational surfaces, their Brauer groups vanish.

For $X = \mathbb{P}(1,1,3)$, the minimal resolution is $\mathbb{F}_3$, and the exceptional curve is the negative section $E \subset \mathbb{F}_3$. If F denotes a fiber of the ruling $\mathbb{F}_3 \to \mathbb{P}^1$, then $F \cdot E = 1$. Hence the map $\operatorname{Pic}(\mathbb{F}_3) \to E^* \cong \mathbb{Z}$, given by $D \mapsto D \cdot E$, is surjective. Since $\operatorname{Br}(\mathbb{F}_3) = 0$, Bright's sequence gives

$$\operatorname{Br}(\mathbb{P}(1,1,3)) = 0.$$

The same argument applies to X_{10}. Let $f : Y \to X_{10}$ be the minimal resolution, and let $E \subset Y$ be the exceptional curve over the unique singular point of type $\frac{1}{3}(1,1)$. The local class group of this singularity is $\mathbb{Z}/3\mathbb{Z}$, and the Weil divisor class $H = \mathcal{O}_{X_{10}}(1)$ maps to a generator of this local class group. Equivalently, if $\widetilde{H}$ denotes the strict transform of a representative of H on Y, then $\widetilde{H} \cdot E$ is congruent to 1 modulo 3. Since $E^2 = -3$, the intersection numbers $\widetilde{H} \cdot E$ and E^2 are coprime. Therefore the map $\operatorname{Pic}(Y) \to E^* \cong \mathbb{Z}$, given by $D \mapsto D \cdot E$, is surjective. Since Y is a smooth rational surface, $\operatorname{Br}(Y) = 0$, and Bright's sequence gives

$$\operatorname{Br}(X_{10}) = 0.$$

Thus the Brauer obstruction in the sense of Karmazyn–Kuznetsov–Shinder does not appear for either $\mathbb{P}(1,1,3)$ or X_{10}. What remains is the geometric compatibility condition: the semiorthogonal decomposition on the minimal resolution must be chosen so that the object $\mathcal{O}_E(-1)$ belongs to the component associated with the exceptional (-3)-curve.

5. Corti–Heuberger cascades and the surface X_{10}

We now specialize the preceding results to the hypersurface X_{10}. The new geometric input is the cascade structure of Corti–Heuberger for del Pezzo surfaces with $\frac{1}{3}(1,1)$ singularities. This input identifies the minimal resolution of X_{10} and turns the abstract existence theorem into an explicit construction.

Corti–Heuberger classify non-smooth del Pezzo surfaces with $\frac{1}{3}(1,1)$ points into qG-deformation families and organize these families into unprojection cascades. A cascade is obtained by blowing up nonsingular points, or equivalently by reversing a sequence of contractions of floating (-1)-curves. Here a floating (-1)-curve is a (-1)-curve contained in the smooth locus of the surface; see [5, Definition 5]. Thus the cascade does not change the analytic type of the $\frac{1}{3}(1,1)$ singularities.

For a fixed number s of $\frac{1}{3}(1,1)$ points, Corti–Heuberger show that the surfaces in the corresponding families are obtained from a finite list of simpler models by blowing up nonsingular points; see [5, Theorem 6 and Corollary 8]. In the case $s = 1$, the relevant statement is that a surface in the family $X_{1,d}$ is the blow-up of $25/3 - d$ nonsingular points on $\mathbb{P}(1,1,3)$. There is one useful nuance in this case. If a sequence first reaches the family $B_{1,16/3}$, Corti–Heuberger show that there is an alternative sequence of four blow-downs of floating (-1)-curves which starts from the same surface and ends at $\mathbb{P}(1,1,3)$; see [5, Remark 9]. Hence, for the one-point families, the cascade can always be read as a sequence ending at $\mathbb{P}(1,1,3)$.

We now specialize to a general degree 10 hypersurface $X_{10} \subset \mathbb{P}(1,2,3,5)$. As recalled earlier, X_{10} has a unique cyclic quotient singularity at $p = [0:0:1:0]$, and this singularity has type $\frac{1}{3}(1,1)$. Weighted adjunction gives

$$K_{X_{10}} \sim \mathcal{O}_{X_{10}}\left(10 - (1+2+3+5)\right) = \mathcal{O}_{X_{10}}(-1).$$

Thus, we have $-K_{X_{10}} \sim \mathcal{O}_{X_{10}}(1)$. Moreover, we obtain

$$K_{X_{10}}^2 = \frac{10}{1 \cdot 2 \cdot 3 \cdot 5} = \frac{1}{3}.$$

Thus X_{10} belongs to the Corti–Heuberger family $X_{1,1/3}$.

Proposition 5.1. *Let $X_{10} \subset \mathbb{P}(1,2,3,5)$ be a general degree 10 hypersurface. Then the minimal resolution of X_{10} is*

$$\widetilde{X}_{10} \cong \mathrm{Bl}_8 \mathbb{F}_3.$$

Moreover, the exceptional curve over the singular point of X_{10} is the strict transform of the negative section of $\mathbb{F}_3$, and it has self-intersection -3.

Proof. Since X_{10} belongs to the family $X_{1,1/3}$, the $s = 1$ cascade of Corti–Heuberger gives

$$\frac{25}{3} - \frac{1}{3} = 8$$

blow-ups from $\mathbb{P}(1,1,3)$ to X_{10}. Equivalently, X_{10} admits a sequence of eight contractions of floating (-1)-curves whose final surface is $\mathbb{P}(1,1,3)$. These curves lie in the smooth locus, so the cascade does not affect the local resolution of the $\frac{1}{3}(1,1)$ point. Since the minimal resolution of $\mathbb{P}(1,1,3)$ is $\mathbb{F}_3$, passing to minimal resolutions gives $\widetilde{X}_{10} \cong \mathrm{Bl}_8 \mathbb{F}_3$. The exceptional curve over the singular point is therefore the strict transform of the negative section of $\mathbb{F}_3$. □

We now record the categorical consequences of this explicit resolution. Let $\sigma : \widetilde{X}_{10} \to \mathbb{F}_3$ be the composition of the eight blow-downs, and let $E_1, \ldots, E_8$ be the corresponding exceptional curves. On $\mathbb{F}_3$, with negative section C_0 and fiber class f, we use the full exceptional collection

$$\langle \mathcal{O}, \mathcal{O}(f), \mathcal{O}(C_0 + 3f), \mathcal{O}(C_0 + 4f) \rangle .$$

By applying Orlov's blow-up formula eight times, we obtain a full exceptional collection on $\widetilde{X}_{10}$ of length 12:

$$\langle \mathcal{O}_{E_1}(-1), \ldots, \mathcal{O}_{E_8}(-1), \sigma^*\mathcal{O}, \sigma^*\mathcal{O}(f), \sigma^*\mathcal{O}(C_0 + 3f), \sigma^*\mathcal{O}(C_0 + 4f) \rangle .$$

Let $\pi : \mathcal{X}_{10} \to X_{10}$ be the canonical stack. Since X_{10} has exactly one singularity of type $\frac{1}{3}(1,1)$, the Ishii–Ueda decomposition adds one exceptional object to the image of $D^b(\mathrm{coh}\, \widetilde{X}_{10})$. Therefore we obtain

$$D^b(\mathrm{coh}\, \mathcal{X}_{10}) = \left\langle \mathcal{E}_p, \Phi(D^b(\mathrm{coh}\, \widetilde{X}_{10})) \right\rangle ,$$

where $\mathcal{E}_p$ is the exceptional object associated with the unique non-special representation at the stacky point.

Theorem 5.2. *The canonical stack $\mathcal{X}_{10}$ has a full exceptional collection of length 13.*

Proof. By Proposition 5.1, the minimal resolution of X_{10} is $\widetilde{X}_{10} \cong \mathrm{Bl}_8 \mathbb{F}_3$. Hence $D^b(\mathrm{coh}\, \widetilde{X}_{10})$ has a full exceptional collection of length 12. The unique singularity of type $\frac{1}{3}(1,1)$ contributes one additional exceptional object through the Ishii–Ueda decomposition. Thus $D^b(\mathrm{coh}\, \mathcal{X}_{10})$ has a full exceptional collection of length $12 + 1 = 13$. □

We also obtain the corresponding decomposition for the singular coarse surface. The KKS decomposition for the contraction $\mathbb{F}_3 \to \mathbb{P}(1,1,3)$ has a component adherent to the negative section C_0. The eight cascade blow-ups occur away from this curve, so the Orlov blow-up components can be added without changing the adherent component. Since the minimal resolution $\widetilde{X}_{10} \to X_{10}$ contracts only the strict transform of C_0, the resulting semiorthogonal decomposition on $\widetilde{X}_{10}$ is compatible with the contraction to X_{10}.

Theorem 5.3. *The singular coarse surface X_{10} admits a semiorthogonal decomposition*

$$D^b(\mathrm{coh}\, X_{10}) = \left\langle D^b(K(3,1)\text{-}mod), F_1, \ldots, F_{10} \right\rangle,$$

where $F_1, \ldots, F_{10}$ are exceptional objects and

$$K(3,1) \cong k[z_1, z_2]/(z_1, z_2)^2.$$

Proof. The KKS decomposition for $\mathbb{P}(1,1,3)$ descends from a semiorthogonal decomposition of $D^b(\mathrm{coh}\, \mathbb{F}_3)$ whose first component is adherent to the negative section C_0. By Proposition 5.1, the minimal resolution of X_{10} is obtained from $\mathbb{F}_3$ by eight blow-ups away from C_0. Orlov's blow-up formula adds eight exceptional objects, and the adherent component containing $\mathcal{O}_{C_0}(-1)$ is unchanged. Hence the resulting semiorthogonal decomposition on $\widetilde{X}_{10}$ is compatible with the contraction $\widetilde{X}_{10} \to X_{10}$. By the descent theorem of Karmazyn–Kuznetsov–Shinder, it descends to a semiorthogonal decomposition of $D^b(\mathrm{coh}\, X_{10})$. The descended adherent component is equivalent to $D^b(K(3,1)\text{-mod})$, and the remaining components are the two exceptional components coming from $\mathbb{P}(1,1,3)$ together with the eight Orlov blow-up components. Thus the decomposition has the stated form. □

6. Further directions: cascades and good models

We finish by explaining how the construction used for X_{10} fits into a more general pattern suggested by the work of Cavey–Prince [4]. The point is that the categorical argument separates into two parts. The first part is global and uses a cascade to describe the minimal resolution as a sequence of blow-ups of an explicit rational surface. The second part is local and uses the Ishii–Ueda decomposition to add exceptional objects indexed by non-special representations of the stabilizer groups.

We first consider the case of one singular point. Let X be a log del Pezzo surface with a single singularity of type $\frac{1}{k}(1,1)$. Cavey–Prince show that the relevant deformation families fit into cascades starting from $\mathbb{P}(1,1,k)$; see [4]. The surfaces in such a cascade are obtained by successive blow-ups of smooth points. Since these blow-ups occur away from the singular point, the analytic type of the singularity remains $\frac{1}{k}(1,1)$ throughout the cascade.

The starting model has a simple resolution. The minimal resolution of $\mathbb{P}(1,1,k)$ is the Hirzebruch surface $\mathbb{F}_k$. The exceptional curve over the singular point is the negative section $C_0 \subset \mathbb{F}_k$, and it satisfies $C_0^2 = -k$. Therefore, if a surface in the cascade is obtained from $\mathbb{P}(1,1,k)$ by blowing up smooth points away from the singular point, then its minimal resolution is obtained from $\mathbb{F}_k$ by the corresponding blow-ups away from C_0.

This already gives the global part of the categorical construction. The surface $\mathbb{F}_k$ has a full exceptional collection, and Orlov's blow-up formula preserves the existence of full exceptional collections under point blow-ups. Hence the minimal resolution of every surface appearing in such a cascade has a full exceptional collection.

It remains to describe the local contribution from the canonical stack. For the singularity $\frac{1}{k}(1,1)$, the Wunram sequence gives the special representations ρ_0 and ρ_1 of $\boldsymbol{\mu}_k$. Hence the non-special representations are $\rho_2, \rho_3, \ldots, \rho_{k-1}$. Thus one singularity of type $\frac{1}{k}(1,1)$ contributes $k-2$ local exceptional objects to the Ishii–Ueda semiorthogonal complement.

The case of several singularities requires a different starting point. One should not start from $\mathbb{P}(1,1,k)$, since that surface has only one singularity. Instead, the initial model should already have the required basket of singularities. For instance, Cavey–Prince describe models with two singularities as hypersurfaces

$$X_{k_1+k_2} \subset \mathbb{P}(1,1,k_1,k_2),$$

with singularities of types $\frac{1}{k_1}(1,1)$ and $\frac{1}{k_2}(1,1)$. In this situation, the cascade should consist of blow-ups of smooth points on this model, so the basket of singularities remains fixed along the cascade.

For example, if one wants a surface with singularities of types $\frac{1}{3}(1,1)$ and $\frac{1}{6}(1,1)$, then the appropriate starting point is not a weighted projective plane. Instead, one starts from a model such as

$$X_9 \subset \mathbb{P}(1,1,3,6),$$

and then performs blow-ups at smooth points. The Ishii–Ueda contribution is determined locally by the two stabilizer groups. The $\frac{1}{3}(1,1)$ point contributes one object, corresponding to ρ_2, while the $\frac{1}{6}(1,1)$ point contributes four objects, corresponding to $\rho_2, \rho_3, \rho_4, \rho_5$. Thus the two singularities contribute five local exceptional objects in total.

This gives the guiding principle for the multi-singularity case. One should first choose a model whose basket already contains the desired singularities, then analyze the cascade by blow-ups of smooth points, and finally add the gerbe-supported Ishii–Ueda objects for each singular point. A general theorem in this direction would require a precise cascade description and a compatibility check for the chosen initial model.

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